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Janus and multifaced supersymmetric theories

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Abstract: We investigate the various properties of Janus supersymmetric Yang-Mills theories. A novel vacuum structure is found and BPS monopoles and dyons are studied. Less supersymmetric Janus theories found before are derived by a simpler method. In addition, we find the supersymmetric theories when the coupling constant depends on two and three spatial coordinates.

Keywords: Gauge-gravity correspondence, Extended Supersymmetry, AdS-CFT Correspondence.

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1. Introduction and conclusion

The AdS-CFT correspondence gives rise to many insights to the conformal field theories [1–3]. The most successful example is the relation between the string theory on $AdS_5 \times S^5$ and the 4-dim $\mathcal{N} = 4$ supersymmetric Yang-Mills theories. The original Janus solution in ref. [4] is a 1-parameter family of dilatonic deformations of AdS_5 space without supersymmetry. This solution turns out to be stable under a large class of perturbations [4–6] and some holographic properties have been explored in refs. [4, 5, 7] The Janus solution is made of two Minkowski spaces joined along an interface so that the dilaton field interpolates two asymptotic values. The CFT dual field theory is suggested to be the deformation of the Yang-Mills theory where the coupling constant changes from one region to another region at 2-dim interface [4, 8].

Further works revealed that one can have supersymmetric Janus geometries with the various supersymmetris and internal symmetries [8–10]. Starting from the 16 supersymmetric Yang-Mills theory, the various deformations of 0, 2, 4, 8 supersymmetries have been found [11]. Especially, the 16 supersymmetric Janus geometries have been found [12–14]. Also other aspects of the Janus solutions have been discussed in ref. [15–19].

Instead of following the detail of the frame work given in ref. [11] where the 6-dim symplectic Majorana fermions are used extensively, we start from the 10-dim supersymmetric Yang-Mills theory where the discussions are quite simple. In this work, we give a simple derivation of the deformation of the 16 supersymmetric Yang-Mills theory.

One could ask whether there is a supersymmetric deformation of the Yang-Mills theory where the coupling constant depends on time too. Indeed there have been several works along this direction [20 − 23]. To maintain some supersymmetry, the time dependency of the coupling constant should accompany the spatial dependency, say $e^2(t + x)$. It turns out that there is no need to correct the Lagrangian or the supersymmetric transformation besides reducing the supersymmetry by 1/2 by imposing a constraint on the supersymmetry parameter spinor.

Starting from 10-dim supersymmetric Yang-Mills theories, one may wonder about the higher dimensional Janus theories. For the simplest case with 8 supersymmetries, one can easily read off from the Lagrangian that such theory can exist in 7-dim spacetime as one needs 3 scalar fields. For the less supersymmetric case one needs more scalar fields, and so lower dimension. Additional spatial dependency of the coupling constant also needs more scalar fields to maintain some supersymmetry. Results in the section 6 and section 7 casshows the maximum spacetime dimension, depending on the cases.

The supersymmetric vacuum of the 8 supersymmetric Janus is governed by the Nahm equation [24]. Besides the usual Coulomb phase, there can be nontrivial vacuum where the nonabelian gauge symmetry is completely broken near the planes where the coupling constant $e^2(z)$ can vanish. In addition, one can have 1/2 BPS magnetic monopoles and charged particles and 1/4 BPS dyons in the Coulomb phase.

In the limit of a sharp interface, one needs various continuity condition on the fields. Especially one can see that there are mirror charges for magnetic monopoles and electrically charged particles in the Coulomb phase. An incident massless wave on a sharp face are partially reflected and partially transmitted without refraction.

In this work we study in detail the properties of 8 supersymmetric Janus Yang-Mills theories, like the vacuum structure and the BPS configurations. In addition, we recapitulate the less supersymmetric Janus theories found in [11]. Then we classify all the supersymmetric deformations of the 16 supersymmetric Yang-Mills theories when the coupling constant depends on the two or three spatial coordinates. These higher dimensional cases tend to have less supersymmetries. We have not explored in the detail the properties of these less supersymmetric theories. There may be some surprises. Nonsupersymmetric geometry with a special higher dimensional Janus type has been worked out [18]. Our work suggests a possibility of supersymmetric Janus geometries where the dilaton field depends on several coordinates.

When one has a theta term which also depends on the coupling constant, one may wonder there can be a supersymmetric theory. For example, the Yang-Mills parts of the Lagrangian can be written as

$$\mathrm{Tr}\frac{1}{4e^2}\left(-F_{\mu\nu}F^{\mu\nu} + \tan\alpha F_{\mu\nu}\tilde{F}^{\mu\nu}\right) , \tag{1.1}$$

where $\tan\alpha = e^2\theta/8\pi^2$. As one can obtain the Janus geometry where both dilaton and axion changes by the $\mathrm{SL}(2,R)$ transformation [13], one expects a supersymmetric Lagrangian with the theta-term. However, we have not found one yet.

Our analysis of Janus theories are done in the classical level. Once the quantum effect is included, one expect the coupling constants to run. It is not clear how to define the infrared limit of the coupling constant. We can choose an arbitrary profile for the coupling constant $e^2(z)$ at the ultraviolet region and maybe the effective coupling constant at the low energy may take a universal profile.

We would like to point out some gap between Janus solution in supergravity and Janus field theory. The maximally supersymmetric Janus solution in supergravity has a limited number of parameters for the dilaton field. This contrasts to the field theory which can have arbitrary profile of coupling constants. The coupling constant profile can be regarded as an ultraviolet profile and the quantum corrections would lead to a change of profile in low energy. However, we do not expect any universal profile at the low energy as the high energy profile can be chosen to be oscillate. Thus we believe that the Janus field theory provides a larger set of theories than those described by the supergravity solution, and would like to find out other alternative origin of Janus field theory. Also, one could ask which is the exactly corresponding CFT for the supersymmetric Janus gravity solution. It would be interesting to learn more about both Janus field theory and gravity solution and their relations.

We worked out the cases with the matter fields. One can start from 6-dim theory with hypermultiplets, 4-dim theory with chiral multiplets, or 3-dim theory with matter multiplets. The detail will appear soon.

The plan of the paper is as follows. In section 2, we review the 8 supersymmetric Janus Yang-Mills theories. In section 3, we study the vacuum structure of this theory. In section 4, we consider the BPS monopoles and dyons in this theory. In section 5, we focus on the sharp interface for the the coupling constant. The image charges for the magnetic monopoles and electric charges are found. The wave propagation and reflection at the interface is studied. In section 6, less supersymmetric Janus Yang-Mills theories are found with four real parameters. In section 7, we find the supersymmetric deformation of the Yang-Mills theories when the coupling constant depends on 2 spacial coordinates. In section 8, we find the supersymmetric deformation in the case where the coupling constant depends on all three spatial coordinates.

2. 8 supersymmetric Janus Lagrangian

The 10-dim supersymmetric Yang-Mills Lagrangian is

$$\mathcal{L}_0 = \frac{1}{4e^2}\mathrm{Tr}\Big(- F^{MN}F_{MN} - 2i\bar{\lambda}\Gamma^M D_M\lambda\Big), \tag{2.1}$$

where $M, N = 0, 1, 2, \ldots, 9$. We use the 10-dim notation for convenience with the gamma matrices Γ^M in the Majorana representation and the gaugino field λ is Majorana and Weyl. The spatial signature is $(- + + + \cdots +)$. The Lagrangian is invariant under the original

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supersymmetric transformation

$$\delta_0 A_M = i\bar{\lambda}\Gamma_M \epsilon\,,\ \delta_0\lambda = \frac{1}{2}\Gamma^{MN}\epsilon F_{MN}, \tag{2.2}$$

where the Weyl-condition on the susy parameter ϵ is

$$\Gamma^{012\cdots 9}\epsilon = \epsilon. \tag{2.3}$$

The spinor ϵ is also a Majorana spinor. As we consider $1+3$ dim spacetime x^0, x^1, x^2, x^3, the remaining spatial gradient $\partial_M = 0$ with $M = 4, 5, \ldots, 9$ and the gauge field A_M become scalar fields ϕ_M with $M = 4, 5, \ldots, 9$. The theory has 16 supersymmetries.

In this work, the coupling constant e^2 can depend on space-time coordinates. The original Lagrangian $\mathcal{L}_0$ transforms as a total derivative under the original supersymmetric transformation δ_0 so that

$$\delta_0 \mathcal{L}_0 = -\partial_\mu \left(\frac{1}{4e^2}\right) \mathrm{Tr}\left(\bar{\lambda}\Gamma^{MN}\Gamma^\mu \epsilon F_{MN}\right). \tag{2.4}$$

Fortunately, one can maintain some of supersymmetries if one corrects the supersymmetric transformation of the gaugino field by $\delta_1\lambda$ and also the Lagrangian by additional terms which depend on the spatial derivatives of the coupling constant. The additional transformation of the original Lagrangian due to $\delta_1\lambda$ would be

$$\delta_1 \mathcal{L}_0 = -\partial_\mu \left(\frac{1}{2e^2}\right) \mathrm{Tr}\left(i\bar{\lambda}\Gamma^\mu \delta_1\lambda - \frac{1}{e^2} i\bar{\lambda}\Gamma^M D_M \delta_1\lambda\right). \tag{2.5}$$

Let us start with the case where the coupling constant e^2 depends only on the $x^3 = z$ coordinate. The coupling constant $e^2(z)$ can be an arbitrary function. The original 16 supersymmetries should be broken to 8 supersymmetries or less [10]. The natural choice of the additional condition on the spinor ϵ compatible with the Weyl condition (2.3) is

$$\Gamma^{3456}\epsilon = \epsilon. \tag{2.6}$$

This condition breaks the number of supersymmetries to 8 and the global SO(6) symmetry which rotates $4, 5, 6, 7, 8, 9$ indices to $\mathrm{SO}(3) \times \mathrm{SO}(3)$, each of which rotates $4, 5, 6$ and $7, 8, 9$ indices respectively.

To cancel some of terms in the zeroth order variation of the original Lagrangian (2.4), one needs to add a correction to the susy transformation of the gaugino field and the corrections to the original Lagrangian. The correction to the original susy transformation (2.2) is

$$\delta_1 A_M = 0\ ,\ \ \delta_1\lambda = e^2 \left(\frac{1}{e^2}\right)' \sum_{a=4,5,6} \Gamma^{3a}\epsilon\phi_a, \tag{2.7}$$

where the prime means d/dz. The correction to the original Lagrangian is made of two parts. The first correction, which depends on the first order in the derivative of the couple constant, is given as

$$\mathcal{L}_1 = \left(\frac{1}{4e^2}\right)' \mathrm{Tr}\Big(i\bar{\lambda}\Gamma^{456}\lambda - 8i\phi_4[\phi_5, \phi_6]\Big). \tag{2.8}$$

The second correction, which is second order in the derivative, is given as

$$\mathcal{L}_2 = -\frac{e^2}{2}\left(\frac{1}{e^2}\right)' \partial_3\Big(\frac{1}{e^2}\mathrm{Tr}\sum_{a=4,5,6}\phi_a^2\Big). \tag{2.9}$$

The total Lagrangian $\mathcal{L} = \mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2$ is invariant under the corrected susy transformation,

$$\begin{aligned}\delta A_M &= (\delta_0+\delta_1)A_M = i\bar{\lambda}\Gamma_M\epsilon,\\ \delta\lambda &= (\delta_0+\delta_1)\lambda = \frac{1}{2}F_{MN}\Gamma^{MN}\epsilon + e^2\left(\frac{1}{e^2}\right)'\Gamma^{3a}\epsilon\phi_a.\end{aligned} \tag{2.10}$$

The susy parameter ϵ is constant in spacetime. There is no requirement on the space dependence of the coupling constant as long as it is smooth.

The total Lagrangian $\mathcal{L} = \mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2$ becomes somewhat simpler with change of the field variables as noted in [11]. We divide $\phi_I,\ I=4,5,\ldots,9$ to two groups so that

$$\tilde{\phi}_a \equiv \frac{1}{e^2}\phi_a,\ (a=4,5,6),\ \ \phi_i=\phi_i\ (i=7,8,9). \tag{2.11}$$

The whole Lagrangian $\mathcal{L}$ becomes

$$\begin{aligned}\mathcal{L} = &\frac{1}{4e^2}\mathrm{Tr}\Big(-F^{\mu\nu}F_{\mu\nu} - 2D^\mu\phi_iD_\mu\phi_i - 2e^4D^\mu\tilde{\phi}_aD_\mu\tilde{\phi}_a\Big)\\ &+\frac{1}{4e^2}\mathrm{Tr}\Big([\phi_i,\phi_j]^2 - 2e^4[\phi_i,\tilde{\phi}_a]^2 + e^8[\tilde{\phi}_a,\tilde{\phi}_b]^2\Big)\\ &-\frac{i}{2e^2}\mathrm{Tr}\Big(\bar{\lambda}\Gamma^\mu D_\mu\lambda - i\bar{\lambda}\Gamma^i[\phi_i,\lambda] - ie^2\bar{\lambda}\Gamma^a[\tilde{\phi}_a,\lambda]\Big)\\ &+\left(\frac{1}{4e^2}\right)'\mathrm{Tr}\Big(\bar{\lambda}\Gamma^{456}\lambda - 8ie^6\tilde{\phi}_4[\tilde{\phi}_5,\tilde{\phi}_6]\Big).\end{aligned} \tag{2.12}$$

The combined susy transformation (2.10) becomes

$$\begin{aligned}\delta A_\mu &= i\bar{\lambda}\Gamma_\mu\epsilon, \qquad \delta\tilde{\phi}_a = \frac{1}{e^2}\bar{\lambda}\Gamma_a\epsilon, \qquad \delta\phi_i = \bar{\lambda}\Gamma_i\epsilon,\\ \delta\lambda &= \Big(\frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu} + e^2D_\mu\tilde{\phi}_a\Gamma^{\mu a} + D_\mu\phi_i\Gamma^{\mu i}\\ &\qquad -ie^2[\tilde{\phi}_a,\phi_i]\Gamma^{ai} - \frac{i}{2}e^4[\tilde{\phi}_a,\tilde{\phi}_b]\Gamma^{ab} - \frac{i}{2}[\phi_i,\phi_j]\Gamma^{ij}\Big)\epsilon.\end{aligned} \tag{2.13}$$

We can choose the gauge group to be any simple Lie group G.

We consider the case where the coupling constant $e^2(z)$ remain positive everywhere except some isolated planes defined by $z=z_r, r=1,2,\ldots p$ where $e^2(z)$ vanishes. While we expect the field ϕ_I to be continuous and differentiable everywhere, we do not expect $\tilde{\phi}_a = \phi_a/e^2$ to be finite and continuous across the zero planes of the coupling constant. This would be an important point in the study of the vacuum structure.

If the coupling $e^2(z)$ is an even function of z, the Lagrangian is symmetric under the following Z_2 transformation

$$z\to -z\,,\ A_z\to -A_z(-z)\,,\ \tilde{\phi}_a\to-\tilde{\phi}_a\,(a=4,5,6)\,,\ \lambda\to\Gamma^{3456}\lambda. \tag{2.14}$$

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On the other hand, the coupling constant $e^2(z)$ can interpolate a strong coupling regime with a weak coupling regime. For example, we can choose the coupling constant profile to be

$$\frac{e^2(z)}{4\pi} = \frac{4\pi}{e^2(-z)}. \tag{2.15}$$

The electric coupling and magnetic coupling constants are exchanged as one crosses the interface. In this case, the spacial reflection (2.14) exchanges the electric and magnetic sectors.

3. Vacuum structure

Let us consider the minimum of the bosonic energy density. At the minimum of the energy, the gauge field strength vanishes and the gauge field A_μ is chosen to be zero in a gauge. One can allow the usual Coulomb phase where the scalar fields ϕ_i and $\tilde{\phi}_a$ are homogeneous and diagonal. The Janus theory may allow additional vacuum structure, as there are corrections to the original Lagrangian. To see this, let us consider the energy density for the field $\tilde{\phi}_a$ while showing only $x^3 = z$ dependence for the simplicity. The bosonic energy density becomes

$$\begin{aligned}\mathcal{E} &= \frac{e^2}{2}\mathrm{Tr}\left((D_3\tilde{\phi}_a)^2 - \frac{e^4}{2}[\tilde{\phi}_a, \tilde{\phi}_b]^2\right) - i(e^4)'\mathrm{Tr}\left(\tilde{\phi}_4[\tilde{\phi}_5, \tilde{\phi}_6]\right) \\ &= \frac{e^2}{2}\mathrm{Tr}\left(D_3\tilde{\phi}_a + \frac{e^2}{2}\epsilon_{abc}i[\tilde{\phi}_b, \tilde{\phi}_c]\right)^2 - i\mathrm{Tr}\left(e^4\tilde{\phi}_4[\tilde{\phi}_5, \tilde{\phi}_6]\right)'.\end{aligned} \tag{3.1}$$

Thus the energy functional is bounded below at zero energy if the boundary term vanishes. The classical vacuum configurations with zero energy satisfy

$$A_\mu = 0, \quad \partial_\mu\phi_i = 0, \quad \partial_{0,1,2}\tilde{\phi}_a = 0, \quad [\phi_i, \phi_j] = 0, \quad [\phi_i, \tilde{\phi}_a] = 0, \tag{3.2}$$

$$D_3\tilde{\phi}_a + \frac{e^2}{2}\epsilon_{abc}i[\tilde{\phi}_b, \tilde{\phi}_c] = 0. \tag{3.3}$$

The last equation is true whenever $e^2 \neq 0$. The vacuum configurations preserve all the supersymmetries, as the gaugino transformation (2.13) becomes

$$\delta\lambda = e^2\Gamma^{0a}\left(D_3\tilde{\phi}_a + \frac{ie^2}{2}\epsilon_{abc}[\tilde{\phi}_b, \tilde{\phi}_c]\Gamma^{3456}\right)\epsilon = 0. \tag{3.4}$$

The contribution of the boundary term to the energy functional is given by

$$e^4(z)\mathcal{F}(z)\Big|_{-\infty}^{+\infty}, \tag{3.5}$$

where

$$\mathcal{F}(z) = -i\mathrm{Tr}\big(\tilde{\phi}_4[\tilde{\phi}_5, \tilde{\phi}_6]\big). \tag{3.6}$$

Using the vacuum equation (3.3), we get

$$\frac{d}{dz}\mathcal{F}(z) = e^2\mathrm{Tr}\Big(- [\tilde{\phi}_4, \tilde{\phi}_5]^2 - [\tilde{\phi}_5, \tilde{\phi}_6]^2 - [\tilde{\phi}_6, \tilde{\phi}_4]^2\Big) \geq 0, \tag{3.7}$$

and so the function $\mathcal{F}(z)$ is non-decreasing in z in the interval where $e^2(z)$ is nonvanishing. Thus the boundary term would not vanish if $e^2(z)$ is nonzero everywhere, and $\mathcal{F}(z)$ is nonzero somewhere. However we can have nontrivial nonabelian vacuum such that the boundary contributions vanish when e^2 vanishes somewhere, including $z = \pm\infty$.

To solve the vacuum equation (3.3), let us introduce a new variable u such that

$$du = e^2(z)dz, \quad \text{or} \quad u = \int_0^z dz\, e^2(z). \tag{3.8}$$

In the gauge $A_z = 0$, the vacuum equation becomes

$$e^2\left(\frac{d\tilde{\phi}_a}{du} + \frac{i}{2}\epsilon_{abc}[\tilde{\phi}_b, \tilde{\phi}_c]\right) = 0. \tag{3.9}$$

When $e^2 \neq 0$, the above equation is the Nahm equation for magnetic monopoles [24]. However at points where $e^2(z) = 0$, the Nahm equations does not need to hold. As before we assume that $e^2(z)$ vanishes at finite number of points z_r, and we divides the $z = x^3$ line into finite number of intervals separated by zero points z_r. The fields $\tilde{\phi}_a$ need not be continuous nor finite at these zero points as long as the original unscaled field ϕ_a is so. Thus we are solving the Nahm at each interval. For each interval between zero coupling constant points z_r, one has to impose the Nahm equations in u variables. In addition we require the contribution of the boundary term to be finite, continuous at z_r, and vanishes at $\pm\infty$.

To be more concrete let us focus on the gauge group SU(2). The general solutions of the Nahm equation can be obtained by using the ansatz,

$$\tilde{\phi}_{3+a} = f_a(u)\frac{\sigma_a}{2} \tag{3.10}$$

with the Pauli matrices σ_a and no sum over the indices $a = 1, 2, 3$. The vacuum equation becomes

$$f_1' = f_2 f_3, \quad f_2' = f_3 f_1, \quad f_3' = f_1 f_2, \tag{3.11}$$

whose solutions are given in terms of the Jacobi elliptic functions, as follows:

$$\begin{aligned}
f_1(u; k, D, u_0) &\equiv -\frac{D\,\mathrm{cn}_k[D(u-u_0)]}{\mathrm{sn}_k[D(u-u_0)]}, \\
f_2(u; k, D, u_0) &\equiv -\frac{D\,\mathrm{dn}_k[D(u-u_0)]}{\mathrm{sn}_k[D(u-u_0)]}, \\
f_3(u; k, D, u_0) &\equiv -\frac{D}{\mathrm{sn}_k[D(u-u_0)]},
\end{aligned} \tag{3.12}$$

where $k \in [0, 1]$ is the elliptic modulus, and two parameters $D \geq 0,\ u_0$ are arbitrary. This solution blows up when sn_k goes to zero. The zeros of $\mathrm{sn}_k(w)$ is $w = 0, 2K(k)$ where $K(k)$ is the complete elliptic integral of the first kind. The function $K(k)$ goes to infinite at the boundary $k = 1$. The above solution in this limit becomes

$$\begin{aligned}
f_1(u; k = 1, D, u_0) &= -\frac{D\cosh(D(u-u_0))}{\sinh(D(u-u_0))}, \\
f_2(u; k = 1, D, u_0) &= f_3(u; k = 1, D, u_0) = -\frac{D}{\sinh D(u-u_0)}.
\end{aligned} \tag{3.13}$$

When $D \neq 0$ nor $K(k) = \infty$, the general solution (3.12) blows up at finite u. If there is no point including infinities where e^2 vanishes, one can see there is no nontrivial vacuum solution.

Let us now consider the case where e^2 vanishes only one point, say at $z = 0$, and remain positive and finite everywhere else. We do not need the detail profile of the coupling constant $e^2(z)$ for our discussion. The parameter u in eq. (3.8) is negative for $z < 0$ and positive for $z > 0$. We have two semi-infinite intervals and so need the above solution (3.13) for these two intervals. We could choose independent parameters for two interval and so the vacuum solution becomes

$$\tilde{\phi}_{3+a} = \begin{cases} f_a(u; k = 1, D_-, u_-)\frac{\sigma_a}{2} & \text{for } z < 0 \\ f_a(u; k = 1, D_+, u_+)\frac{\sigma_a}{2} & \text{for } z > 0 \end{cases}, \tag{3.14}$$

where $u_- > 0, u_+ < 0$. The range of two parameters $u_\pm$ is chosen so that $\tilde{\phi}_a$ does not diverge anywhere. If we have chosen $u_- = 0$, we would have divergent contribution to the boundary term at 0_- as $\phi_a \sim 1/(e^2(t)t)\sigma_a$ near $z = 0_-$. The asymptotic values of $\tilde{\phi}$ at the spatial infinity becomes

$$\tilde{\phi}_a(z = \pm\infty) = -\delta_{a4} D_\pm \frac{\sigma_1}{2}. \tag{3.15}$$

Not only the asymptotic value $D_\pm$ can be different, they can vanish. Thus, one can have nontrivial vacuum even in the symmetric phase. The above solution (3.14) becomes abelian in asymptotic region ($z = \pm\infty$) but nonabelian close to the zero plane $z = 0$. The SU(2) gauge symmetry is completely broken near the wall but becomes abelian when $D_\pm \neq 0$ or fully restored when $D_\pm = 0$ at the boundaries $z = \pm\infty$.

When there are more planes where $e^2(z)$ vanishes, one can have a richer vacuum structure. For each finite interval between zeros, the full general solution (3.12) will play a role. The above solution (3.14) becomes abelian in asymptotic region ($z = \pm\infty$) but nonabelian close to the zero planes $z = z_r$. The SU(2) gauge symmetry is completely broken near the wall but becomes abelian or fully restored at the boundaries. There are several parameters characterizing the vacuum, besides the global SU(2) rotation of three scalar fields $\tilde{\phi}_a$. The detailed physics in a given vacuum is intriguing but will not be pursued in this work.

4. BPS objects

The BPS configurations are those which respect some supersymmetries. Let us consider the supersymmetric transformation (2.10) of the gaugino field. In each vacuum one can study the BPS configurations. The supersymmetry preserved by the BPS configurations should be compatible with the original supersymmetric condition, $\Gamma^{3456}\epsilon = \epsilon$. We will consider the following two conditions on the supersymmetric parameter, ϵ;

$$\Gamma^{1234}\epsilon = \alpha\epsilon, \quad \Gamma^{07}\epsilon = \beta\epsilon, \tag{4.1}$$

where $\alpha = \pm 1, \beta = \pm 1$. The above relations imply that $\Gamma^{1256}\epsilon = -\alpha\epsilon$, $\Gamma^{1289}\epsilon = \beta\epsilon$, and $\Gamma^{5689}\epsilon = \alpha\beta\epsilon$. We could impose only one condition and then the configurations would be 1/2 BPS. If we impose both conditions, the configurations would be 1/4 BPS.

One may wonder whether there are other possible BPS conditions. As the fields are Majorana, we cannot introduce, for example, the projection $\Gamma^{12}\epsilon = i\epsilon$. Other possible projections like $\Gamma^{1256}\epsilon = \epsilon$ or $\Gamma^{1289}\epsilon = \epsilon$ are allowed. But these conditions would lead to the reduction of the selfdual Yang-Mills equation to 2-spatial direction, which does not have any obvious nontrivial smooth solution. The above BPS conditions (4.1) are those for magnetic monopoles and charged W-bosons in non-Janus case and might imply nontrivial BPS configurations even in the Janus case.

The supersymmetric transformation (2.10) of the gaugino field can be expressed as

$$\begin{aligned}\delta\lambda = {} & \Gamma^{p0}(F_{p0} - D_p\phi_7\Gamma^{07})\epsilon + e^2\Gamma^{0a}(D_0\tilde{\phi}_a + i[\phi_7,\tilde{\phi}_a]\Gamma^{07})\epsilon + \sum_{i=8,9}\Gamma^{0i}(D_0\phi_i + i[\phi_7,\phi_i]\Gamma^{07})\epsilon \\ & +\Gamma^{12}(F_{12} - e^2D_3\tilde{\phi}_4\Gamma^{1234} + e^4 i[\tilde{\phi}_5,\tilde{\phi}_6]\Gamma^{1256} + i[\phi_8,\phi_9]\Gamma^{1289})\epsilon + \Gamma^{23}(F_{23} - e^2D_1\tilde{\phi}_4\Gamma^{1234})\epsilon \\ & +\Gamma^{31}(F_{31} - e^2D_2\tilde{\phi}_4\Gamma^{1234})\epsilon + e^2\Gamma^{15}(D_1\tilde{\phi}_5 + D_2\tilde{\phi}_6\Gamma^{1256})\epsilon + e^2\Gamma^{25}(D_2\tilde{\phi}_5 - D_1\tilde{\phi}_6\Gamma^{1256})\epsilon \\ & +e^2\Gamma^{35}(D_3\tilde{\phi}_5 + ie^2[\tilde{\phi}_6,\tilde{\phi}_4]\Gamma^{3456})\epsilon + e^2\Gamma^{36}(D_3\tilde{\phi}_6 + ie^2[\tilde{\phi}_4,\tilde{\phi}_5]\Gamma^{3456})\epsilon \\ & +\Gamma^{18}(D_1\phi_8 + D_2\phi_9\Gamma^{1289})\epsilon + \Gamma^{28}(D_2\phi_8 - D_1\phi_9\Gamma^{1289})\epsilon \\ & +\Gamma^{38}(D_3\phi_8 - ie^2[\tilde{\phi}_4,\phi_9]\Gamma^{3489})\epsilon + \Gamma^{39}(D_3\phi_9 + ie^2[\tilde{\phi}_4,\phi_8]\Gamma^{3489})\epsilon \\ & +e^2\Gamma^{58}(-i[\tilde{\phi}_5,\phi_8] - i[\tilde{\phi}_6,\phi_9]\Gamma^{5689})\epsilon + e^2\Gamma^{59}(-i[\tilde{\phi}_5,\phi_9] + i[\tilde{\phi}_6,\phi_8]\Gamma^{5689})\epsilon \\ & +D_0\phi_7\Gamma^{07}\epsilon. \end{aligned} \tag{4.2}$$

The susy transformation $\delta\lambda$ would vanish for the BPS configurations. After using the BPS conditions (4.1), $\delta\lambda = 0$ if all terms vanish individually. (It would be interesting to show that it is also a necessary condition.) Let us consider the magnetic 1/2 BPS equation with $\alpha = 1$. We require all terms vanish with $\beta = \pm 1$. The nontrivial part of the equations for the 1/2 BPS configurations with $\Gamma^{1234}\epsilon = \epsilon$ is made of

$$\begin{aligned} & F_{12} - e^2D_3\tilde{\phi}_4 - ie^4[\tilde{\phi}_5,\tilde{\phi}_6] = 0, \qquad F_{23} - e^2D_1\tilde{\phi}_4 = 0, \quad F_{31} - e^2D_2\tilde{\phi}_4 = 0, \\ & D_3(\tilde{\phi}_5 + i\tilde{\phi}_6) - e^2[\tilde{\phi}_4,\tilde{\phi}_5 + i\tilde{\phi}_6] = 0, \quad (D_1 + iD_2)(\tilde{\phi}_5 + i\tilde{\phi}_6) = 0. \end{aligned} \tag{4.3}$$

This is a mixed form of the Nahm equation for the vacuum and the old BPS equation for magnetic monopoles. The 1/4 BPS dyonic magnetic monopole with $\beta = 1$ can found also. The additional BPS equation for dyons in the gauge $A_0 = \phi_7$ and the ansatz $\phi_8 = \phi_9 = 0$ is simply the Gauss law,

$$-D_p\left(\frac{1}{e^2}D_p\phi_7\right) + e^2[\tilde{\phi}_a,[\tilde{\phi}_a,\phi_7]] = 0. \tag{4.4}$$

In the abelian Coulomb phase, $\tilde{\phi}_5 = \tilde{\phi}_6 = 0$ and the above BPS equations become somewhat simpler. (Of course it would be interesting to find whether there is nontrivial BPS configurations lying beyond the ansatz $\phi_= \phi_9 = 0$.)

For simplicity, let us consider the energy bound in the abelian Coulomb vacuum. Keeping only nontrivial terms, we express the energy functional as

$$\mathcal{H} = \int d^3x\, \frac{1}{2e^2}\mathrm{Tr}\Big((F_{p0} - D_p\phi_7)^2 + e^4(D_0\tilde{\phi}_4 - i[\phi_7,\tilde{\phi}_4])^2 + (B_p - e^2D_p\tilde{\phi}_4)^2\Big) + Q_e + Q_m, \tag{4.5}$$

where $B_p = \frac{1}{2}\epsilon_{pkl}F^{kl}$ and

$$Q_e = \int d^3x \partial_p \mathrm{Tr}\left(\frac{1}{e^2} F_{p0}\phi_7\right), \quad Q_m = \int d^3x \partial_p \mathrm{Tr}(B_p \tilde{\phi}_4), \tag{4.6}$$

are the electric and magnetic energy contributions, respectively. In the Janus field theory, the coupling constant e^2 depending on the spatial coordinates and so it is much harder to solve the BPS equations even for a single magnetic monopole. Magnetic monopoles are topologically characterized in usual abelian vacuum, but it is not clear whether it is so in a nonabelian vacuum.

5. A sharp interface

5.1 BPS monopoles and point electric charge

Suppose the coupling constant $e^2(z)$ changes from one value to another at a sharp interface so that

$$e(z) = \begin{cases} e_1 \text{ for } z > 0 \\ e_2 \text{ for } z < 0 \end{cases}. \tag{5.1}$$

Such a limit can be obtained by shrinking the interface region to a plane. As there is no additional source term at the interface, we get the continuity conditions of the various fields. The continuous ones are the following fields and their covariant derivatives:

$$\begin{aligned} &F_{01}, F_{02}, F_{12}, \frac{F_{03}}{e^2}, \frac{F_{23}}{e^2}, \frac{F_{31}}{e^2}, \\ &\tilde{\phi}_a, D_1\tilde{\phi}_a, D_2\tilde{\phi}_a, e^2 D_3\tilde{\phi}_a, && a = 4,5,6 \\ &\phi_i, D_1\phi_i, D_2\phi_i, \frac{D_3\phi_i}{e^2}, && i = 7,8,9. \end{aligned} \tag{5.2}$$

Thus naturally we can assume the continuity condition for the infinitesimal gauge function Λ and its derivatives $D_1\Lambda, D_2\Lambda, D_3\Lambda/e^2$.

For simplicity, we consider the SU(2) gauge theory which is broken spontaneous to U(1) subgroup by the Higgs expectation values at the vacuum,

$$\left\langle \tilde{\phi}_4 \right\rangle = \tilde{v}\frac{\sigma_3}{\sqrt{2}}. \tag{5.3}$$

Note that the expectation value of the original field variable $\phi_4 = e^2\tilde{\phi}_4$ makes a jump at the interface. The diagonal components of the fields will be massless and off-diagonal fields will be massive. Let us try to solve the BPS equations in the abelian limit where the nonabelian core size vanishes. For a single monopole at $z = a > 0$, we get the BPS configuration

$$B_i = e^2 D_i\tilde{\phi}_4 = \begin{cases} \frac{(x,y,z-a)}{r_+^3} + \frac{e_1^2-e_2^2}{e_1^2+e_2^2}\frac{(x,y,z+a)}{r_-^3} & , z > 0 \\ \frac{2e_2^2}{e_1^2+e_2^2}\frac{(x,y,z-a)}{r_+^3} & , z < 0 \end{cases}. \tag{5.4}$$

Here we dropped the group factor $\sigma_3/\sqrt{2}$ for the simplicity. The continuous scalar field $\tilde{\phi}_4$ becomes

$$\tilde{\phi}_4 = \begin{cases} \tilde{v} - \frac{1}{e_1^2 r_+} - \frac{e_1^2-e_2^2}{e_1^2(e_1^2+e_2^2)}\frac{1}{r_-} & , z>0 \\ \\ \tilde{v} - \frac{2}{e_1^2+e_2^2}\frac{1}{r_+} & , z<0 \end{cases} . \tag{5.5}$$

The total magnetic flux near $z = a$ is 4π as expected. In the region $z > 0$ where the monopole exists, the total field is that of the magnetic monopole and that of the mirror image at $z = -a$. The total magnetic flux 4π at the spacial infinity consists of the $4\pi e_1^2/(e_1^2 + e_2^2)$ flux from the $z > 0$ hemisphere and the $4\pi e_2^2/(e_a^2 + e_2^2)$ flux from the $z < 0$ hemisphere.

Let us now turn off the $\tilde{\phi}_4$ expectation value and turn on the new expectation value

$$\langle \phi_7 \rangle = u\frac{\sigma_3}{\sqrt{2}}. \tag{5.6}$$

Let us put an unit electric charge at point $(x, y, z) = (0, 0, a > 0)$. The Gauss law is simplified as $\nabla_i(E_i/e^2) = \rho_e$ whose spatial integration is quantized as integer. Ignoring the nonabelian core and dropping the group factor $\sigma_3/\sqrt{2}$ for the simplicity, we get the BPS point charge configuration as

$$E_i = D_i\phi_7 = \begin{cases} \frac{e_1^2}{4\pi}\left(\frac{(x,y,z-a)}{r_+^3} + \frac{-e_1^2+e_2^2}{e_1^2+e_2^2}\frac{(x,y,z+a)}{r_-^3}\right) & , z>0 \\ \frac{e_1^2}{4\pi}\frac{2e_2^2}{e_1^2+e_2^2}\frac{(x,y,z-a)}{r_+^3} & , z<0 \end{cases} . \tag{5.7}$$

where $r_\pm^2 = x^2 + y^2 + (z \mp a)^2$. Note that $E_1, E_2, E_3/e^2$ are continuous along the interface. The continuous scalar field becomes

$$\phi_7 = \begin{cases} u - \frac{e_1^2}{4\pi}\left(\frac{1}{r_+} + \frac{-e_a^2+e_2^2}{e_1^2+e_2^2}\frac{1}{r_-}\right) & , z>0 \\ \\ u - \frac{2e_1^2e_2^2}{4\pi(e_1^2+e_2^2)}\frac{1}{r_+} & , z<0 \end{cases} . \tag{5.8}$$

The total electric charge is the unity near $z = 1$ and remains so at the spatial infinity as it is the sum $e_2^2/(e_1^2 + e_2^2), (z > 0)$ and $e_1^2/(e_1^2 + e_2^2), z < 0$.

5.2 Reflection and transmission of massless waves

Let us consider now a massless wave propagating toward the interface (5.1) of the two coupling constant from $z > 0$ region. The fields and their derivatives in (5.2) should be continuous cross the interface $z = 0$. Let us use the vector notation $\mathbf{E} = (F_{10}, F_{20}, F_{30})$, and $\mathbf{B} = (F_{23}, F_{31}, F_{12})$ for the electromagnetic fields. A part of the incident wave will be reflected and the rest may get refracted or transmitted. Let us call the electromagnetic field of the incident wave to be $\mathbf{E}, \mathbf{B}$, the reflected wave to be $\mathbf{E}'', \mathbf{B}''$ and the transmitted wave to be $\mathbf{E}', \mathbf{B}'$. The continuity equations at $z = 0$ are

$$\begin{aligned}
\big(\mathbf{E} + \mathbf{E}'' - \mathbf{E}'\big) \times \hat{z} &= 0, \\
\big(\mathbf{B} + \mathbf{B}'' - \mathbf{B}\big) \cdot \hat{z} &= 0, \\
\left(\frac{\mathbf{E} + \mathbf{E}''}{e_1^2} - \frac{\mathbf{E}'}{e_2^2}\right) \cdot \hat{z} &= 0, \\
\left(\frac{\mathbf{B} + \mathbf{B}''}{e_1^2} - \frac{\mathbf{B}'}{e_2^2}\right) \times \hat{z} &= 0.
\end{aligned} \tag{5.9}$$

The space-time dependence waves would be $e^{-iwt+\mathbf{k}\cdot\mathbf{x}}$, $e^{-iwt+\mathbf{k}''\cdot\mathbf{x}}$, and $e^{-iwt+\mathbf{k}'\cdot\mathbf{x}}$ for the incident, reflected, and transmitted waves, respectively. The wave equation at each region and the above continuity equations imply that

$$w = |\mathbf{k}| = |\mathbf{k}''| = |\mathbf{k}'|, \;\; \mathbf{k} = \mathbf{k}', \;\; (\mathbf{k} + \mathbf{k}') \times \hat{z} = 0. \tag{5.10}$$

Thus the transmitted wave is not refracted at all. After taking out the space-time dependence, we can express the electric fields of the reflected and transmitted waves in terms of the electric field of the incident wave. While the relation will depends on whether the wave has transverse electric (that is, transverse to the incident plane defined by $\mathbf{k}$ and $\hat{z}$), or transverse magnetic, both cases has the same relation between the magnitude of the electric field at $z = 0$, as $E_0'' = rE_0, E_0' = tE_0$ where the reflection and transmission magnitudes are

$$r = \left|\frac{e_1^2 - e_2^2}{e_1^2 + e_2^2}\right|, \;\; t = \frac{2e_2^2}{e_1^2 + e_2^2}. \tag{5.11}$$

For the vector, one should be careful about the sign, which can be easily fixed by the continuity equations. The same reflection and transmission magnitudes apply to the scalar fields $\phi_i, i = 7, 8, 9$. For the scalar field $\tilde{\phi}_a$, the same reflection magnitude applies but the transmission magnitude becomes $t = 2e_1^2/(e_1^2 + e_2^2)$.

6. Additional Susy breaking Janus

In this section we are still interested in the case where the coupling constant $e^2(z)$ depends only on one spatial coordinate. We can impose additional constraints on the susy parameters ϵ which is compatible with what we have already imposed. There are several of them and so one can break the susy to 1/4 or 1/8, which introduces some free parameters in the interface Lagrangian. We easily recover the results in ref. [11]. As shown in this reference, our study exhaust all possibilities with some supersymmetries. Thus the minimum one will have two supersymmetries for the case where the coupling constant depends only on one spatial direction $e^2(z)$. The compatible conditions including one in (2.6) on the 10-dim Majorana Weyl spinor ϵ are

$$\Gamma^{3456}\epsilon = \epsilon\,, \;\Gamma^{3489}\epsilon = -\epsilon\,, \;\Gamma^{3597}\epsilon = -\epsilon\,, \;\Gamma^{3678}\epsilon = -\epsilon. \tag{6.1}$$

As the product of the above four conditions is an identity, there are only three independent conditions, breaking the supersymmetry to 1/8th or two supersymmetries.

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To cancel $\delta_0\mathcal{L}_0$ in (2.5), we choose the first correction to the Lagrangian to be

$$\begin{aligned}\mathcal{L}_1 = &\left(\frac{1}{4e^2}\right)' \mathrm{Tr}\Big(i\bar{\lambda}(c_0\Gamma^{456} - c_1\Gamma^{489} - c_2\Gamma^{597} - c_3\Gamma^{678})\lambda \\ &-8i\Big(c_0\phi_4[\phi_5,\phi_6] - c_1\phi_4[\phi_8,\phi_9] - c_2\phi_5[\phi_9,\phi_7] - c_3\phi_6[\phi_7,\phi_8]\Big)\Big),\end{aligned} \tag{6.2}$$

where real parameters c_i satisfy

$$c_0 + c_1 + c_2 + c_3 = 1. \tag{6.3}$$

The correction to the susy transformation (2.2) is

$$\begin{aligned}\delta_1\lambda = e^2\left(\frac{1}{e^2}\right)' \Gamma^3 \Bigg(&c_0 \sum_{a=4,5,6} \Gamma^a\phi_a + c_1 \sum_{a=4,8,9} \Gamma^a\phi_a \\ &+c_2 \sum_{a=5,9,7} \Gamma^a\phi_a + c_3 \sum_{a=6,7,8} \Gamma^a\phi_a\Bigg)\epsilon.\end{aligned} \tag{6.4}$$

The second order correction of the Lagrangian is chosen to be

$$\begin{aligned}\mathcal{L}_2 = -\frac{e^2}{2}\left(\frac{1}{e^2}\right)' \partial_3\Bigg(\frac{1}{e^2}\mathrm{Tr}\Bigg(&c_0 \sum_{a=4,5,6}\phi_a^2 + c_1 \sum_{a=4,8,9}\phi_a^2 + c_2 \sum_{a=5,9,7}\phi_a^2 \\ &+c_3 \sum_{a=6,7,8}\phi_a^2\Bigg)\Bigg) + \frac{e^2}{2}\left(\frac{1}{e^2}\right)'^2 \mathrm{Tr}\Big((c_0+c_1)(c_2+c_3)(\phi_4^2+\phi_7^2) \\ &+(c_0+c_2)(c_1+c_3)(\phi_5^2+\phi_8^2)+(c_0+c_3)(c_1+c_2)(\phi_6^2+\phi_9^2)\Big).\end{aligned} \tag{6.5}$$

As noted in ref. [11], notice that when $c_0 = c_1 = c_2 = c_3 = 1/4$, there is an enhanced global symmetry SU(3) with 1/8 supersymmetry. For $c_0 = c_1 = 1/2$ and $c_2 = c_3 = 0$, there is 1/4 supersymmetry with enhanced global symmetry $\mathrm{SO}(2) \times \mathrm{SO}(2)$.

7. Multifaced interfaces in 2,3 dimensions

7.1 $e^2(y,z)$ case

Let us first start with the case where the coupling constant $e^2(y,z)$ depends on only two coordinates. There exist only two independent, modulo rotation, sets of the compatible supersymmetry conditions which are

$$\Gamma^{2789}\epsilon = \epsilon, \qquad \Gamma^{3456}\epsilon = \epsilon, \tag{7.1}$$

$$\Gamma^{2459}\epsilon = -\epsilon, \qquad \Gamma^{3456}\epsilon = \epsilon. \tag{7.2}$$

Each condition breaks the supersymmetry to 1/4. One can break the supersymmetry further to 1/8 by imposing both conditions (7.1) and (7.2) at the same time. Also one can impose additional compatible supersymmetry condition

$$\Gamma^{2567}\epsilon = -\epsilon,\ \Gamma^{3456}\epsilon = \epsilon. \tag{7.3}$$

Imposing these three mutually independent and compatible conditions (7.1), (7.2), (7.3) breaks the supersymmetry to the minimal one 1/16. Note that the conditions (7.2) and (7.3) are related by a rotation. These three conditions imply

$$\Gamma^{2648}\epsilon = \Gamma^{3489}\epsilon = \Gamma^{3597}\epsilon = \Gamma^{3678}\epsilon = -\epsilon. \tag{7.4}$$

These conditions include the conditions (6.1) in the previous section.

We extend the result in the previous section. To cancel $\delta_0\mathcal{L}_0$, we choose choose the first order correction to the Lagrangian to be

$$\begin{aligned}\mathcal{L}_1 =& \partial_2\left(\frac{1}{4e^2}\right)\mathrm{Tr}\Big(i\bar{\lambda}(b_0\Gamma^{789}-b_1\Gamma^{567}-b_2\Gamma^{648}-b_3\Gamma^{459})\lambda \\ & -8i\Big(b_0\phi_7[\phi_8,\phi_9]-b_1\phi_5[\phi_6,\phi_7]-b_2\phi_6[\phi_4,\phi_8]-b_3\phi_4[\phi_5,\phi_9]\Big)\Big) \\ & +\partial_3\left(\frac{1}{4e^2}\right)\mathrm{Tr}\Big(i\bar{\lambda}(c_0\Gamma^{456}-c_1\Gamma^{489}-c_2\Gamma^{597}-c_3\Gamma^{678})\lambda \\ & -8i\Big(c_0\phi_4[\phi_5,\phi_6]-c_1\phi_4[\phi_8,\phi_9]-c_2\phi_5[\phi_9,\phi_7]-c_3\phi_6[\phi_7,\phi_8]\Big)\Big),\end{aligned} \tag{7.5}$$

where real parameters b_i, c_i satisfy

$$b_0+b_1+b_2+b_3=1,\ c_0+c_1+c_2+c_3=1. \tag{7.6}$$

The correction to the supersymmetric transformation (2.6) is

$$\begin{aligned}\delta_1\lambda =& e^2\partial_2\left(\frac{1}{e^2}\right)\Gamma^2\left(b_0\sum_{a=7,8,9}\Gamma^a\phi_a+b_1\sum_{a=5,6,7}\Gamma^a\phi_a+b_2\sum_{a=6,4,8}\Gamma^a\phi_a+b_3\sum_{a=4,5,9}\Gamma^a\phi_a\right)\epsilon \\ & +e^2\partial_3\left(\frac{1}{e^2}\right)\Gamma^3\left(c_0\sum_{a=4,5,6}\Gamma^a\phi_a+c_1\sum_{a=4,8,9}\Gamma^a\phi_a+c_2\sum_{a=5,9,7}\Gamma^a\phi_a+c_3\sum_{a=6,7,8}\Gamma^a\phi_a\right)\epsilon.\end{aligned} \tag{7.7}$$

The additional correction to the Lagrangian is made of

$$\begin{aligned}\mathcal{L}_2 =& -\frac{e^2}{2}\partial_2\left(\frac{1}{e^2}\right)\partial_2\left(\frac{1}{e^2}\mathrm{Tr}\left(b_0\sum_{a=7,8,9}\phi_a^2+b_1\sum_{a=5,6,7}\phi_a^2+b_2\sum_{a=6,4,8}\phi_a^2+b_3\sum_{a=4,5,9}\phi_a^2\right)\right) \\ & -\frac{e^2}{2}\partial_3\left(\frac{1}{e^2}\right)\partial_3\left(\frac{1}{e^2}\mathrm{Tr}\left(c_0\sum_{a=4,5,6}\phi_a^2+c_1\sum_{a=4,8,9}\phi_a^2+c_2\sum_{a=5,9,7}\phi_a^2+c_3\sum_{a=6,7,8}\phi_a^2\right)\right).\end{aligned} \tag{7.8}$$

One needs additional correction to the Lagrangian which are made of mixed terms,

$$\begin{aligned}\mathcal{L}_3 =& \frac{e^2}{2}\left(\partial_2\left(\frac{1}{e^2}\right)\right)^2\mathrm{Tr}\Big((b_0+b_1)(b_2+b_3)(\phi_7^2+\phi_4^2)+(b_0+b_2)(b_1+b_3)(\phi_8^2+\phi_5^2) \\ & +(b_0+b_3)(b_1+b_2)(\phi_9^2+\phi_6^2)\Big)+\frac{e^2}{2}\left(\partial_3\left(\frac{1}{e^2}\right)\right)^2\mathrm{Tr}\Big((c_0+c_1)(c_2+c_3)(\phi_4^2+\phi_7^2) \\ & +(c_0+c_2)(c_1+c_3)(\phi_5^2+\phi_8^2)+(c_0+c_3)(c_1+c_2)(\phi_6^2+\phi_9^2)\Big)\end{aligned}$$

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$$-\left(\partial_2\partial_3\left(\frac{1}{e^2}\right)-e^2\partial_2\left(\frac{1}{e^2}\right)\partial_3\left(\frac{1}{e^2}\right)\right)\mathrm{Tr}\Big((b_0+b_1+c_0+c_1-1)\phi_4\phi_7$$
$$+(b_0+b_2+c_0+c_2-1)\phi_6\phi_8+(b_0+b_3+c_0+c_3-1)\phi_7\phi_9\Big). \quad (7.9)$$

The total Lagrangian $\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3$ is invariant under the corrected supersymmetric transformation. Note that when $e^2 = f(y)g(z)$ so that it is factorizable, the last term vanishes. When $b_a = c_a = 1/4$ for all $a = 0, 1, 2, 3$, there is SO(3) symmetry which rotates $\phi_{4,5,6}$ and $\phi_{7,8,9}$ at the same time. We think our Lagrangian is the most general on in the 2-dim case.

7.2 $e^2(x, y, z)$ case

When the coupling constant depends on all three coordinates $e^2(x, y, z)$, there are two independent susy conditions

$$\Gamma^{1467}\epsilon = \Gamma^{2475}\epsilon = \Gamma^{3456}\epsilon = \epsilon, \quad (7.10)$$
$$\Gamma^{1458}\epsilon = \Gamma^{2468}\epsilon = \Gamma^{3478}\epsilon = \epsilon. \quad (7.11)$$

One can break either susy further to 1/16 by imposing the above two conditions (7.10) and (7.11) together. These two conditions imply $\Gamma^{1234}\epsilon = \Gamma^{5678}\epsilon = -\epsilon$. We choose the first correction to the Lagrangian to be

$$\mathcal{L}_1 = \partial_1\left(\frac{1}{4e^2}\right)\mathrm{Tr}\left(i\bar{\lambda}\Gamma^4(a_1\Gamma^{67}+a_2\Gamma^{58})\lambda - 8i\phi_4(a_1[\phi_6,\phi_7]+a_2[\phi_5,\phi_8])\right)$$
$$+\partial_2\left(\frac{1}{4e^2}\right)\mathrm{Tr}\left(i\bar{\lambda}\Gamma^4(b_1\Gamma^{75}+b_2\Gamma^{68})\lambda - 8i\phi_4(b_1[\phi_7,\phi_5])+b_2[\phi_6,\phi_8])\right)$$
$$+\partial_3\left(\frac{1}{4e^2}\right)\mathrm{Tr}\left((i\bar{\lambda}\Gamma^4(c_1\Gamma^{56}+c_2\Gamma^{78})\lambda - 8i\phi_4(c_1[\phi_5,\phi_6]+c_2[\phi_7,\phi_8])\right), \quad (7.12)$$

where

$$a_1+a_2 = 1,\ b_1+b_2 = 1,\ c_1+c_2 = 1. \quad (7.13)$$

We choose the correction for the susy transformation to be

$$\delta_1\lambda = e^2\partial_1\left(\frac{1}{e^2}\right)\Gamma^1\left(a_1\sum_{a=4,6,7}\phi_a\Gamma^a + a_2\sum_{4,5,8}\phi_a\Gamma^a\right)\epsilon$$
$$+e^2\partial_2\left(\frac{1}{e^2}\right)\Gamma^2\left(b_1\sum_{a=4,7,5}\phi_a\Gamma^a + b_2\sum_{a=4,6,8}\phi_a\Gamma^a\right)\epsilon$$
$$+e^2\partial_3\left(\frac{1}{e^2}\right)\Gamma^3\left(c_1\sum_{a=4,5,6}\phi_a\Gamma^a + c_2\sum_{a=4,7,8}\phi_a\Gamma^a\right)\epsilon. \quad (7.14)$$

The additional Lagrangian becomes

$$\begin{aligned}\mathcal{L}_2 = &-\frac{e^2}{2}\partial_1\left(\frac{1}{e^2}\right)\partial_1 \mathrm{Tr}\big(a_1(\phi_4^2+\phi_6^2+\phi_7^2)+a_2(\phi_4^2+\phi_5^2+\phi_8^2)\big)\\ &-\frac{e^2}{2}\partial_2\left(\frac{1}{e^2}\right)\partial_2 \mathrm{Tr}\big(b_1(\phi_4^2+\phi_5^2+\phi_7^2)+b_2(\phi_4^2+\phi_6^2+\phi_8^2)\big)\\ &-\frac{e^2}{2}\partial_3\left(\frac{1}{e^2}\right)\partial_3 \mathrm{Tr}\big(c_1(\phi_4^2+\phi_5^2+\phi_6^2)+c_1(\phi_4^2+\phi_7^2+\phi_8^2)\big). \qquad (7.15)\end{aligned}$$

The final mixed correction to the Lagrangian is

$$\begin{aligned}\mathcal{L}_3 = &\frac{e^2}{2}\left(a_1a_1\left(\partial_1\left(\frac{1}{e^2}\right)\right)^2+b_1b_2\left(\partial_2\left(\frac{1}{e^2}\right)\right)^2+c_1c_2\left(\partial_3\left(\frac{1}{e^2}\right)\right)^2\right)\mathrm{Tr}\left(\phi_5^2+\phi_6^2+\phi_7^2+\phi_8^2\right)\\ &+\left(\partial_1\partial_2\left(\frac{1}{e^2}\right)-e^2\partial_1\left(\frac{1}{e^2}\right)\partial_2\left(\frac{1}{e^2}\right)\right)\mathrm{Tr}\Big((a_2-b_1)\phi_5\phi_6+(a_1-b_1)\phi_7\phi_8\Big)\\ &+\left(\partial_2\partial_3\left(\frac{1}{e^2}\right)-e^2\partial_2\left(\frac{1}{e^2}\right)\partial_3\left(\frac{1}{e^2}\right)\right)\mathrm{Tr}\Big((b_2-c_1)\phi_6\phi_7+(b_1-c_1)\phi_5\phi_8\Big)\\ &+\left(\partial_3\partial_1\left(\frac{1}{e^2}\right)-e^2\partial_3\left(\frac{1}{e^2}\right)\partial_1\left(\frac{1}{e^2}\right)\right)\mathrm{Tr}\Big((c_2-a_1)\phi_7\phi_5+(c_1-a_1)\phi_6\phi_8\Big). \qquad (7.16)\end{aligned}$$

Note that $a_2 - b_1 = b_2 - a_1 = (a_2 + b_2 - a_1 - b_1)/2$ and $a_1 - b_1 = (a_1 - b_1 - a_2 + b_2)/2$. The last three terms vanish if $e^2(x, y, z)$ has the factorizable spatial dependency. When $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1/2$, we have SO(4) symmetry which rotates $\phi_{5,6,7,8}$. If the coupling constant depends only on the radial variable $e^2(\sqrt{x^2+y^2+z^2})$, there will be a spatial rotational symmetry. Our analysis on the constraint on the spinor is the most general and so our Lagrangian is the most general Lagrangian in 3-dim case.

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A. Moving Janus

Let us consider the space-time dependent coupling constant $e^2(x+t)$. The supersymmetric condition on the constant spinor parameter is

$$\Gamma^{01}\epsilon = \epsilon.$$

Under the infinitesimal supersymmetric transformation, the original Lagrangian transforms as in eq. (2.4). Since $\partial_0 e^2 = \partial_1 e^2$ and $\bar{\epsilon}(\Gamma^0 + \Gamma^1) = 0$, the Lagrangian is invariant under 1/2 of 16 supersymmetries satisfying the above condition. We can mix this time-dependent Janus with other Janus, preserving some supersymmetry if the supersymmetry conditions are compatible.

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